Semiclassical transition probabilities for interacting oscillators

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Abstract

Semiclassical transition probabilities characterize transfer of energy between "hard" and "soft" modes in various physical systems. We establish the boundary problem for singular euclidean solutions used to calculate such probabilities. Solutions are found numerically for a system of two interacting quartic oscillators. In the double-well case, we find numerical evidence that certain regular *minkowskian* trajectories have approximate stopping points or, equivalently, are approximately periodic. This property leads to estimates of tunneling excitation probabilities in that system and suggests that similar estimates may be possible in other systems with tunneling.

1 Introduction

Very little is known at present about semiclassical transition probabilities in systems with more than one degree of freedom. Let x denote one of (possibly many) coordinate variables and $|n\rangle$ be the n-th energy eigenstate. Then, a typical semiclassical matrix element is $\langle n_2|x|n_1\rangle$ for $|n_2 - n_1| \gg 1$. In what follows we always assume $n_2 > n_1$. This matrix element appears when one calculates the probability of exciting a system originally in state $|n_1\rangle$ to state $|n_2\rangle$ by applying a high-frequency external force proportional to x. Similar matrix elements appear in other problems where there is a transfer of energy between "hard" and "soft" modes. The states $|n\rangle$ then refer to the "soft" subsystem. In elementary particle physics, a typical problem of this sort is calculation of cross sections for production of many low-momentum particles in collisions of two high-momentum ones. Electroweak baryon number non-conservation at high energies is an example that was extensively discussed recently (for most recent reviews see ref.[1]) and there are other interesting cases as well [2].

In a system with one degree of freedom, semiclassical transition probabilities can be calculated by the method described in the Landau-Lifshitz textbook [3]. A natural generalization of this method to many degrees of freedom is achieved by reinterpreting it in terms of singular classical solutions [4]. In ref.[5] we have found some approximate singular classical solutions for self-interacting field theories in four dimensions. We have shown how they can be used to reproduce the factorial asymptotics of cross sections for multiparticle production at relatively low energies and how the crossover to a high-energy regime may occur. This approach was pursued further in ref.[6].

There are several problems preventing a full-scale application of the method of singular solutions to field theory. (i) There is no formal derivation of the expression for transition probability used in [4, 5]; consequently, some interpretation problems

arise as to exactly what quantity is being calculated. (ii) No *exact* singular solutions of the required form are known in systems with more than one degree of freedom (except when a reduction to one degree of freedom occurs) and it is not clear what are the existence and uniqueness conditions for such solutions. (iii) In general, one expects that exact singular solutions are accessible only numerically, and the appropriate numerical procedure needs to be formulated.

The present work is an attempt to get at least partial resolution of these problems using the simplest non-trivial case of two interacting non-linear oscillators as an example. In Sect.2 we describe a derivation of the double functional integral representation for the following transition probability

$$W(E_1, E_2) = \sum_{a,b} \left| \langle b | \mathcal{P}_{E_2} x \mathcal{P}_{E_1} | a \rangle \right|^2 , \tag{1}$$

where the sum is over a complete system of states and $\mathcal{P}_E$ is the projector onto the subspace of a given energy E. We then show that in the semiclassical approximation, the transition probability (1) is saturated by a singular solution on a certain contour in the complex time plane. We will comment specifically on the appearance of time dependence in the originally time-independent problem (1). In Sect.3 we present the results of a numerical search for singular classical solutions in a system of two interacting quartic oscillators. We do not attempt a rigorous analytic resolution of the problem (ii) above but rather present numerical evidence for the existence and uniqueness of singular solutions under specified boundary conditions. The results of Sect.3 refer to the case of a single-well potential. The double-well case, which is closer to the problem of electroweak baryon number violation, is considered in Sect.4. We present there numerical evidence that the *minkowskian* solution that starts with zero velocity at a turning point of a periodic instanton (which is a periodic *euclidean* solution [7]) has an approximate stopping point or, in other words, is approximately

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periodic. As we illustrate in Sect.4, this property leads to straightforward numerical estimates of tunneling excitation probabilities in that double-well system. Similar estimates may be possible in other systems with tunneling, maybe even in field theories. Concluding Sect.5 contains a discussion of the results.

2   Semiclassical formula for transition probability

The projector $\mathcal{P}_E$ appearing in (1) has a simple representation, given in [7], only when the system reduces to a collection of non-interacting harmonic oscillators. In field theory, this usually happens at large positive and negative times when an initial configuration dissociates into free particles. A system of a few degrees of freedom, however, typically never gets out of the non-linear regime. To be able to use the representation of ref.[7] for $\mathcal{P}_E$ in the case of two interacting oscillators, we need to approach the original problem in the following way. Two interacting oscillators represent a particle confined by a potential to some region on a two-dimensional plane. Let us couple this particle to electromagnetic field by giving it some small charge. Then, an excited state of the particle will eventually decay into photons, for which our formula for the projector is applicable. For a highly excited state, this decay can be viewed as an almost classical electromagnetic radiation by the oscillating particle. So, the coupling to electromagnetic field does not spoil the semiclassical nature of our problem. Moreover, because the radiation takes place in real time, the action on the corresponding part of the solution does not contribute to the exponential factor in the probability (1). Thus, making the electromagnetic coupling arbitrarily small, we make the probability arbitrarily close to that in the system without electromagnetic coupling, even though at large times the two systems look entirely different.

With this modification in mind, we can use the formula for $\mathcal{P}_E$ at sufficiently large

times. On the other hand, the states in eq.(1) are taken at some fixed moment of time $t = t_0$. To proceed, we should define the state $\mathcal{P}_{E_1}|a\rangle$ at $t = -\infty$ and the state $\mathcal{P}_{E_2}|b\rangle$ at $t = \infty$, where we know how to do that, then evolve these states to $t = t_0$ and use the result in eq.(1). As usual, it is convenient to define initial and final states in the interaction representation. Then, eq.(1) takes the form

$$W(E_1, E_2) = \sum_b \left| \langle b(\infty)|\mathcal{P}_{E_2} S(\infty, t_0) x S(t_0, -\infty) \mathcal{P}_{E_1}|a(-\infty)\rangle \right|^2 , \qquad (2)$$

where $S(\infty, t_0)$ is the "half of the S-matrix" operating from $t = t_0$ to $t = \infty$. Thus, in order to use the known expression for the projector we introduce time dependence in the originally time-independent problem. This may seem to be an additional complication but in fact it is just the opposite. Instead of calculating difficult overlap integrals of wave functions, we find the exponential suppression that is present in $W(E_1, E_2)$ simply from the imaginary part of the action of a classical solution.

In the leading semiclassical approximation, we are interested only in the exponential factor in the probability, so the operator x in eq.(2) is inessential. (Of course, without this operator the *pre-exponent* will vanish.) The exponents of the two halves of the S-matrix in (2) add up to that of the whole S-matrix. We can now proceed in a rather close analogy with the calculation of ref.[7]. The main difference with that calculation is in the contour in the complex time plane, on which a classical solution is to be found. When we consider a transition probability rather than a transition amplitude, we actually need a contour that consists of two parts symmetric with respect to the real axis. In the present case, we use the contour shown in Fig.1 together with its reflection into the lower half-plane. Unlike the contour used in [7], the contour of Fig.1 has both euclidean and anti-euclidean parts, in addition to minkowskian ones. A classical solution along the contour shown in Fig.1 describes a transition from a state of energy E_1 at $t = -\infty + iT_1/2$ to a state of energy E_2 at $t = +\infty + iT_2/2$.
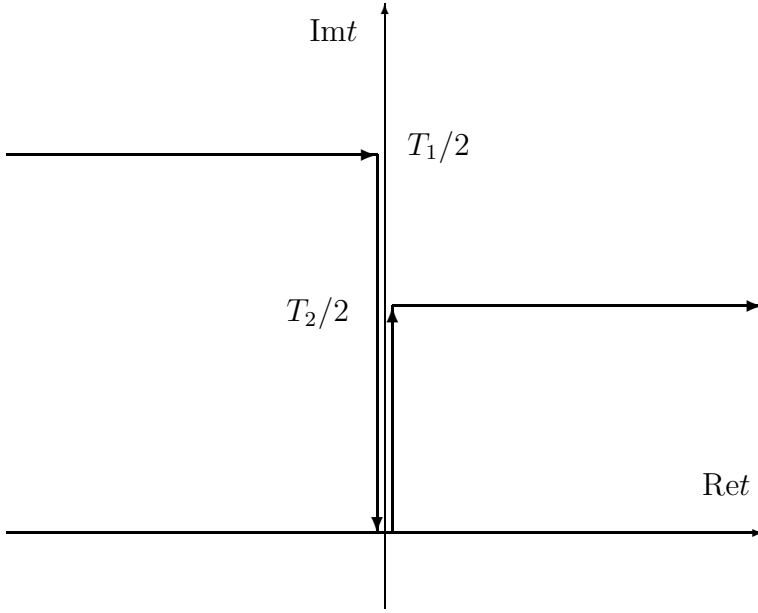
Figure 1: Contour in the complex time plane, on which we look for a real classical solution.

This solution is necessarily singular because the initial and final states have different energies. By convention, the singularity is positioned at $t = 0$.

It is important to realize how crucial is the fact that we consider a total probability and therefore sum over $|a\rangle$ and $|b\rangle$ in eq.(2). If we were to consider probability of transition between certain $|a\rangle$ and $|b\rangle$, we would have to impose the corresponding boundary conditions on the real axis, at $t = -\infty$ and $t = +\infty$. Shifting the points at infinity into the complex plane would then require an adjustment in the boundary values of the fields, leading to complex solutions. A complete system of states, on the other hand, can be inserted anywhere in the complex time plane, so by considering the total probability (2) we allow ourselves to restrict attention to real solutions.*

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* The question remains, which we did not attempt to resolve in this paper, whether additional singularities that a solution can have in the complex time plane are crossed when we deform the contour into the shape shown in Fig.1. One can verify, however, that no such singularities have to

The derivation of the functional-integral representation for the probability (2) using the representations for the S-matrix and the projector $\mathcal{P}_E$ given in [7] is now completely straightforward and we omit the lengthy intermediate expressions. The leading semiclassical result for the transition probability has the form

$$W(E_1, E_2) \sim \exp\left(E_1 T_1 - E_2 T_2 - 2S\right) \equiv \exp(-Q) , \qquad (3)$$

where S is the net euclidean action acquired on the euclidean and anti-euclidean parts of the contour of Fig.1. The action S corresponds to a solution that at $t = iT_1/2$ starts from a turning point with energy E_1, follows euclidean evolution to a singularity at $t = 0$, then follows anti-euclidean evolution to a turning point with energy E_2 at $t = iT_2/2$. The actions corresponding to the euclidean and anti-euclidean evolutions separately are infinite but they come with opposite signs and when added together give a finite net amount S.

The minkowskian parts of the contour do not contribute to eq.(3). Therefore, in applications of eq.(3) we can forget we ever needed to couple our system to the photon "bath" because it is always possible to choose the coupling to photons so small that it does not influence the euclidean and anti-euclidean parts of the evolution.

As we will see in the next section, there are infinitely many singular solutions of the form described above, corresponding to infinitely many possibilities to choose the initial turning point at $t = iT_1/2$. However, there is a preferred one, for which Q in (3) attains the smallest possible value at given E_1 and E_2. This solution saturates the semiclassical transition probability and determines the states at $t = -\infty + iT_1/2$ and $t = +\infty + iT_2/2$. It also determines T_1 and T_2 for given E_1 and E_2.

An analogous calculation in field theory will have to impose a momentum cutoff

be crossed in the case of a single symmetric quartic oscillator where singular solutions can be easily found analytically.

that specifies what modes are considered "soft". This is not an ambiguity of the method because such a cutoff has to be built in any procedure of identifying "soft" particle production in experiment. One might think then that the transition probability is saturated by a solution for which the final state $|b\rangle$ has particles only in a few "hardest" of the remaining "soft" modes, because this allows for smaller multiplicity. However, it is not so clear what the actual answer is: the tendency towards smaller multiplicity has to compete with the tendency to have a spatially localized coherent field configuration, which enhances the probability by achieving stronger non-linearity. Our computations for two oscillators described in Sect.3 show that the tendency towards stronger non-linearity is indeed rather effective in distributing energy between "harder" and "softer" modes in the final state.

3   Numerical results

We now turn to numerical results for the case of two interacting quartic oscillators. The hamiltonian is

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + V(x,y) \, , \tag{4}$$

where the potential $V(x,y)$ is

$$V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}\omega^2 y^2 + \frac{\lambda}{4}\left(x^4 + 6x^2 y^2 + y^4\right) \, . \tag{5}$$

The particular choice of the ratios of couplings in (5) is motivated by the property that the system (4)-(5) is equivalent to the $\lambda\phi^4$ theory on a space "lattice" consisting of two sites. The euclidean and anti-euclidean parts of classical evolution can be viewed as mechanical motions in the potential equal to $-V(x,y)$.

Instead of the coordinates x and y it is convenient to use two other variables – the polar angle ϕ in the (x,y) plane and the value of the potential V. The euclidean

part of a singular solution starts with zero velocity at some angle ϕ_i and $V = E_1 \neq 0$, and the anti-euclidean part stops at angle ϕ_f and $V = E_2 > E_1$. The main question is how we match these two parts at the singularity.

To formulate the matching condition, consider the following limiting procedure. Let us choose some large value of the potential $\Lambda \gg E_1, E_2$. The euclidean trajectory starting with zero velocity at $\phi = \phi_i$, $V = E_1$ will reach, in some time $T_1(\Lambda)$, the value $V = \Lambda$ at some angle $\phi = \phi_\Lambda$. The euclidean action accumulated on this trajectory is some $S_1(\Lambda)$. We now look for the trajectory that starts with zero velocity at some angle $\phi_f(\Lambda)$ and $V = E_2$ and reaches, in some time $T_2(\Lambda)$, exactly the same point $V = \Lambda$, $\phi = \phi_\Lambda$ as the first one. Mechanical intuition suggests that such trajectory always exists and our numerical results also show so. The corresponding euclidean action is some $S_2(\Lambda)$. Due to anharmonicity, both trajectories reach infinite values of V in finite times, in other words, limits

$$T_{1,2} = \lim_{\Lambda \to \infty} T_{1,2}(\Lambda) \tag{6}$$

always exist. A far more subtle property is the existence of limits

$$S = \lim_{\Lambda \to \infty} \left(S_1(\Lambda) - S_2(\Lambda) \right) , \tag{7}$$

$$\phi_f = \lim_{\Lambda \to \infty} \phi_f(\Lambda) . \tag{8}$$

If these limits exist, they define the singular solution corresponding to given E_1, E_2 and a given initial configuration ϕ_i. We then scan among all initial configurations of energy E_1 and find the one that leads to the smallest suppression factor in (3). To summarize, the euclidean and anti-euclidean parts of a solution are matched at some large value of the potential energy Λ and then limit $\Lambda \to \infty$ is taken in the hope that the quantities (7)-(8) remain stable. In what follows, we present numerical evidence in favor of the existence of the limits (7)-(8).

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Numerically, matching at finite Λ is simplified by the following property of eu-clidean trajectories, which we found empirically. Suppose there are three trajectories starting with $V = E$ and zero velocity at angles ϕ_1, ϕ_2 and ϕ_3 and passing through $V = \Lambda$ at angles Φ_1, Φ_2 and Φ_3, respectively. The property is that if $\phi_1 < \phi_2 < \phi_3$, then $\Phi_1 < \Phi_2 < \Phi_3$. This "monotony" allows us to use the familiar root finding methods of bisections and secants, to find the trajectory that starts at $V = E_2$ and passes $V = \Lambda$ at a specified value of the angle.

For our computations we chose the following values of the parameters: $\omega = 1.5$, $\lambda = 0.1$, $E_1 = 0.0001$. Such a small value of E_1 means that the system is excited out of its ground state, which is the case of main interest in many applications. We consider a set of different values of the final state energy E_2 in order to obtain the dependence of the suppression factor (3) on the excitation energy.

In a numerical computation, it is of course impossible to make Λ arbitrarily large. Let us estimate the error introduced by the finiteness of Λ. In the Landau-Lifshitz formula for a quartic oscillator, with $V(x) = x^2/2 + gx^4/4$, the absolute value of the error in $Q/2$ introduced by the finiteness of Λ is

$$\sqrt{2} \int_{x_\Lambda}^{\infty} \left((V(x) - E_1)^{1/2} - (V(x) - E_2)^{1/2} \right) dx = \frac{E_2 - E_1}{(g\Lambda)^{1/4}} + \mathcal{O}(\Lambda^{-3/4}) . \qquad (9)$$

Guided by eq.(9), we estimate the same error for our system (assuming the limits (7)-(8) exist) as

$$\frac{E_2 - E_1}{(\lambda_{\text{eff}}\Lambda)^{1/4}} \leq \frac{E_2 - E_1}{(\lambda\Lambda)^{1/4}} , \qquad (10)$$

where λ_{eff} is an effective coupling constant for a given region of angles, and the inequality follows from the fact that the smallest anharmonicity is along the $x = 0$ and $y = 0$ axes, and this corresponds to $\lambda_{\text{eff}} = \lambda$.

If we want the error (10) to be smaller than some ϵ, it is sufficient to take Λ equal

to or larger than

$$\Lambda_\epsilon = \frac{1}{\lambda} \left(\frac{E_2 - E_1}{\epsilon} \right)^4 .$$ (11)

Large values of Λ dictate the need for high accuracy in solving the euclidean evolution equations and matching euclidean and anti-euclidean parts of a solution, in order to insure that numerical errors coming from those sources do not add up to quantities larger than ϵ. Suppose the numerical error we make in solutions for $x(t)$ and $y(t)$ is some δ. The abbreviated actions $S_{1,2}(\Lambda) - E_{1,2}T_{1,2}(\Lambda)/2$ of the euclidean and anti-euclidean parts of a solution are found according to

$$S_{1,2}(\Lambda) = 2 \int_0^{T_{1,2}(\Lambda)/2} \{V[x_{1,2}(t), y_{1,2}(t)] - E_{1,2}\}dt .$$ (12)

Apart from solutions staying on the $x = 0$ and $y = 0$ axes, for which our system reduces to that of one degree of freedom, every euclidean solution eventually reaches a region where the absolute values of both x and y are large. This is the region where the main error in $S_{1,2}(\Lambda)$ comes from. In such a region, the system (4) with our particular choice of the potential, eq.(5), is effectively separable in terms of variables $\xi = (x+y)/\sqrt{2}$ and $\eta = (x-y)/\sqrt{2}$, so the main error in $S_{1,2}(\Lambda)$ is the sum of errors in corresponding abbreviated actions. The errors in ξ and η are not larger than $\sqrt{2}\delta$. Then, for example, the error in the potential for ξ, V_ξ, is $\sqrt{2}\delta\partial V_\xi/\partial\xi$ and the error in the abbreviated action for ξ is estimated as

$$2\sqrt{2}\delta \int^{T/2} \frac{dV_\xi}{d\xi}dt = 2\delta \int^{\xi_\Lambda} \frac{(dV_\xi/d\xi)\,d\xi}{(V_\xi - E_\xi)^{1/2}}$$
$$= 4\delta\sqrt{V_\xi(\xi_\Lambda)} + \mathcal{O}(\delta/\sqrt{V_\xi(\xi_\Lambda)}) .$$ (13)

(Strictly speaking, some numerical error comes also from determination of $T_{1,2}$ but that can be easily made much smaller than (13).) Even though both $V_\xi(\xi_\Lambda)$ and $V_\eta(\eta_\Lambda)$ are large, only one of them is of order (and hence essentially equal to) Λ. So, the numerical error in each of the abbreviated actions (12), produced by the error δ

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Λ	$0.5 \times \Lambda_\epsilon$	Λ_ϵ	$2.5 \times \Lambda_\epsilon$	$5 \times \Lambda_\epsilon$	$10 \times \Lambda_\epsilon$
$Q/2$	23.05	23.06	23.08	23.09	23.10
ϕ_f	1.40979841	1.40979916	1.40979996	1.40980045	1.40980085

Table 1: Values of half of the exponent Q of eq.(3) and the final state angle ϕ_f for $E_2 = 15$, initial state angle $\phi_i = \pi/6$ and different values of the matching energy Λ. $\Lambda_\epsilon = 5.06 \times 10^9$.

in the solution, is estimated as $4\delta\sqrt{\Lambda}$. If we want the corresponding error in $Q/2$ to be much smaller than ϵ, we have to make δ much smaller than

$$\delta_\epsilon(\Lambda) = \frac{\epsilon}{8\sqrt{\Lambda}} \; . \tag{14}$$

For the smallest admissible Λ, eq.(11), this becomes

$$\delta_\epsilon(\Lambda_\epsilon) = \frac{\epsilon^3\sqrt{\lambda}}{8(E_2 - E_1)^2} \; . \tag{15}$$

We see that even moderate improvements in the desired accuracy ϵ of $Q/2$ require considerable improvements in the accuracy of the solution. The required accuracy of the solution is also higher for transitions between states that are more separated in energy.

Our computations were done with $\epsilon = 0.1$. The equations of motion were solved and euclidean and anti-euclidean parts were matched to accuracy more than an order of magnitude better than (14). Table 1 illustrates the existence of the limits (7)-(8) for $E_2 = 15$ and $\phi_i = \pi/6$. Note that for $\Lambda \geq \Lambda_\epsilon$, the value of $Q/2$ does not change with Λ within the accuracy ϵ, in correspondence with our estimates. We remind that each value of $Q/2$ in Table 1 is obtained as a difference between large (divergent at $\Lambda \to \infty$) abbreviated actions $S_1 - E_1 T_1/2$ and $S_2 - E_2 T_2/2$ acquired on the euclidean and anti-euclidean parts of the solution. For example, for $\Lambda = 5 \times V_\epsilon$, the values of these abbreviated actions are of order 6×10^7.

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ϕ_i	$\sim 10^{-19}$	0.008	0.017	0.028	0.041	0.058	0.083	0.129	0.231	0.537
ϕ_f	0	$\pi/20$	$\pi/10$	$3\pi/20$	$\pi/5$	$\pi/4$	$3\pi/10$	$7\pi/20$	$2\pi/5$	$9\pi/20$
$Q/2$	25.2	23.0	21.9	21.3	21.1	21.0	21.2	21.6	22.2	23.1

Table 2: Several values of the initial angle and the corresponding values of the final angle and $Q/2$ for $E_2 = 15$. Estimated error in $Q/2$ is less than 0.1.

E_2	10	15	25	50
$Q/2$	15.4	21.0	30.9	51.8

Table 3: Values of $Q/2$ determining suppression factors (3) for initial state energy $E_1 = 0.0001$ and various final state energies E_2. Estimated error in $Q/2$ is less than 0.1.

Next we consider values of the exponent Q for fixed E_2 and different values of the initial angle ϕ_i. Results for $E_2 = 15$ are presented in Table 2.[†] We see that Q has a minimum near $\phi_f = \pi/4$. The corresponding value $Q/2 = 21.0$ determines the suppression factor (3) for $E_2 = 15$.

Note that the preferred final state is not purely the "harder" y-component but a state where x and y are comparable, which achieves stronger non-linearity. This tendency persists to larger values of ω. For example, at $\omega = 3$, $E_2 = 10$, very few y-quanta need to be produced but the system prefers ϕ_f near $3\pi/10$ rather than $\phi_f = \pi/2$.

The minimum in $Q/2$ becomes more pronounced as the final state energy E_2 increases. Table 3 gives such minimal values of $Q/2$ for various values of E_2. The suppression increases with the excitation energy, as may be intuitively expected.

[†] These results were actually obtained by finding ϕ_i for a set of ϕ_f equally spaced between 0 and $\pi/2$. Equal spacing of ϕ_f, rather than of ϕ_i, happens to give more points in the most interesting region of angles where Q is close to its minimum.

4 Tunneling probabilities

In the previous sections, we considered transition probabilities, for which we summed over all initial and final states of given energies (or, more precisely, states with energies distributed in a small interval near given ones – microcanonical states). For potentials with two or more local minima, at energies less than the height of the barrier, we may be interested separately in the probability of a tunneling process, when a system originally distributed near energy E_1 in one well gets excited to the state near energy E_2 in another. If $E_1 = E_2$, tunneling probabilities for microcanonical initial states can be calculated using regular euclidean solutions – periodic instantons [7]. When $E_1 \neq E_2$, however, we again have to use singular solutions.

In the case of a single degree of freedom, the relevant singular solution is easily found (see for example ref.[6]). Let us assume $E_1 < E_2$. The periodic instanton corresponding to energy E_2 has two turning points x_2 and x_3, one of them, x_2, in the well where the system is originally located. If we use x_2, with zero velocity, as an initial condition for *minkowskian* evolution, the resulting (real) minkowskian trajectory will stop again at a different point x_1 in the same well. This new turning point x_1, which still has energy E_2, can now be used as the final point of the anti-euclidean segment of a singular euclidean solution whose euclidean segment starts at some x_0 with energy E_1. The whole trajectory from x_0 to x_3 comprises the singular solution that describes the tunneling process. The corresponding suppression exponent is the sum of the exponent Q, eq.(3), acquired from x_0 to x_1, and the suppression exponent associated with the periodic instanton.

The stopping of the minkowskian trajectory at x_1 is a special feature of systems with one degree of freedom. When there are several degrees of freedom, velocity has several components which in general will never vanish simultaneously. Further, there

seems to be no reason why they should vanish simultaneously (except at the initial point) for the minkowskian trajectory starting at the turning point of a periodic instanton. However, a numerical "experiment" shows that in a double-well system of two interacting quartic oscillators (see below) this in fact *nearly* happens. In other words, the minkowskian solution starting at the turning point of a periodic instanton is approximately periodic.

Because the periodicity of the minkowskian solution is not exact, the method described above for the case of one degree of freedom cannot be literally applied to several degrees of freedom. Most likely, to generalize this method to several degrees of freedom, one has to consider complex singular solutions. The approximate periodicity is still of value, however, because it allows us to construct a purely real *approximate* singular solution describing a tunneling transition and use it to *estimate* the probability of that transition.

Specifically, consider the hamiltonian (4) with the following double-well potential,

$$V(x, y) = -\frac{1}{2}x^2 + \frac{1}{2}\omega^2 y^2 + \frac{\lambda}{4}\left(x^4 + 6x^2 y^2 + y^4\right) - \alpha y . \tag{16}$$

The term $-\alpha y$ was added to avoid the trivial situation when stopping occurs simply because $y = 0$ on the whole trajectory. Below we present results for $\omega = 1.5$, $\lambda = 0.01$, $\alpha = 1$ and $E_2 = -15$ (minima of the potential (16) have negative energy). Similar results were obtained for other values of the parameters and for other quartic double-well potentials. The approximate stopping property is less pronounced for potentials of higher order.

We found it convenient to characterize configurations in the left well by their polar angles ϕ in the polar system in which the origin is placed at $x = -1/\lambda^{1/2}$, $y = 0$. (This would be the location of the left minimum of the potential at $\alpha = 0$.) The turning point of the periodic instanton for the quoted values of the parameters is $\phi = 0.0817$.

Figure 2: (a) Absolute value of velocity on the minkowskian trajectory starting with zero velocity at a turning point of a periodic instanton. (b) The same for an arbitrarily chosen initial point.

The absolute value of velocity $v = (\dot{x}^2 + \dot{y}^2)^{1/2}$ on the minkowskian trajectory starting at this point with zero velocity is plotted as a function of time in Fig.2a. We see that the absolute value of the velocity almost reaches zero. Numerically, the minimum absolute value of the velocity (not counting zero value at $t = 0$) in this case is about 3×10^{-4}, which is at least hundred times smaller than its "natural" value. For comparison, Fig.2b shows the same quantity for the minkowskian trajectory starting with zero velocity at arbitrarily chosen point $\phi = 0.01$. Here the absolute value of velocity does not approach zero (except at zero time) and is clearly aperiodic.

We can use the approximate stopping point of a minkowskian solution as the

E_2	-24	-15	-10	-5	-1	-0.25
$Q_{\text{tun}}/2$	87.6	68.5	59.4	50.8	44.3	43.0

Table 4: Values of $Q_{\text{tun}}/2$ determining estimated suppression factors (10) for the potential (9), initial state energy $E_1 = -24$ and various final state energies E_2. Estimated numerical error in $Q_{\text{tun}}/2$ is less than 0.1.

final state ϕ_f for a real singular euclidean solution similar to those found in Sect.3. Because the euclidean solution reaches exactly zero velocity at this point while the minkowskian solution does so only approximately, the trajectory comprised of the two will not be an exact solution. Still, it is useful for estimates of tunneling probabilities in our system. We thus describe the tunneling process as consisting of two stages: first, the system is excited from energy E_1 to energy E_2 while staying in the same well, then it tunnels at energy E_2. The tunneling probability is then estimated as

$$W_{\text{tun}}(E_1, E_2) \sim \exp(-Q + E_2 T' - S') \equiv \exp(-Q_{\text{tun}}) , \tag{17}$$

where Q is associated with the singular euclidean solution in the same way as in eq.(3), S' is the action of the periodic instanton and T' is its period. Table 4 shows values of $Q_{\text{tun}}/2$ for the potential (16) with the same values of ω, λ and α as above, $E_1 = -24$ and various values of E_2. For reference, the energy of the saddle point separating two minima is $E_0 = -0.222$. We see that the suppression decreases as this energy is approached, though it does not go away completely.

5 Conclusion

Our aim in this work was to introduce the boundary problem for singular euclidean solutions and to gain evidence for its correctness by doing numerical calculations for a system of two interacting non-linear oscillators. For two oscillators with a quartic

double-well potential, we also presented numerical evidence that the *minkowskian* trajectories starting with zero velocity from turning points of periodic instantons (which are periodic *euclidean* solutions) have approximate stopping points or, equivalently, are approximately periodic. This property allowed us to construct real approximate singular solutions describing tunneling transitions and use them to estimate the probabilities of such transitions.

One may try to extend the latter approach to other systems, including field theories. The accuracy to which the approximate stopping property holds will differ for different systems, and so will the accuracy of estimates based on it. However, within that accuracy, this approach gives an in principle straightforward method, essentially identical to the method used for a single degree of freedom, of numerical estimates of tunneling excitation rates in systems with many degrees of freedom.

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